Program summary, $N_f = 10$

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Lattice action

Consider a four-dimensional Euclidean lattice with spacing a. Our lattice action S can be split into gauge and fermionic parts, $S = S_g + S_f$. For the gauge part, we will use the standard unimproved Wilson gauge action

$$S_g = -\frac{\beta}{N_c}\sum_P \mathrm{Re\ Tr}\, U_P = -\frac{\beta}{4N_c}a^4\sum_x \mathrm{Tr}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + O(a^6) \tag{1}$$

which reduces to the standard Yang-Mills action in the continuum limit if we identify $\beta = 2N_c/g_0^2$. $\hat{F}_{\mu\nu}$ is a lattice version of the usual field strength tensor. The sum is over plaquettes:

$$\sum_P \mathrm{Re\ Tr}\ U_P = \sum_x \sum_{\mu<\nu} \mathrm{Re\ Tr}\ \left[U_\nu(x)U_\mu(x+\hat{\nu}) \times U_\nu^\dagger(x+\hat{\mu}+\hat{\nu})U_\mu^\dagger(x+\hat{\mu})\right]$$

where the notation $U_\mu(x)$ indicates a gauge link from the lattice site x to the site $x+\hat{\mu}$. The figure to the right shows a typical plaquette.

For our fermions, we employ the "clover" action, which uses Wilson-type fermions along with a dimension 5 operator included to explicitly cancel $O(a)$ lattice artifacts:

$$S_f = a^4\sum_x \overline{\psi}(x)(D+m_0)\psi(x) + a^5 c_{sw}\sum_x \overline{\psi}(x)\frac{i}{4}\sigma^{\mu\nu}\hat{F}_{\mu\nu}\psi(x) + O(a^6) \tag{2}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, and the parameters m_0 and c_{sw} are free.

In the continuum limit, $a^4\sum_x \to \int d^4x$ and we recover the usual continuum action. Terms of $O(a^5)$ and higher in the action will be suppressed by powers of the lattice spacing a, and will thus vanish in the continuum; these terms are generically called "lattice artifacts."

The second term in S_f above is included explicitly to cancel these artifacts; its coefficient is tuned to cancel $O(a)$ contributions that are generated on the lattice. Dimension five operators are forbidden in S_g by gauge invariance, so $O(a)$ artifacts do not exist.

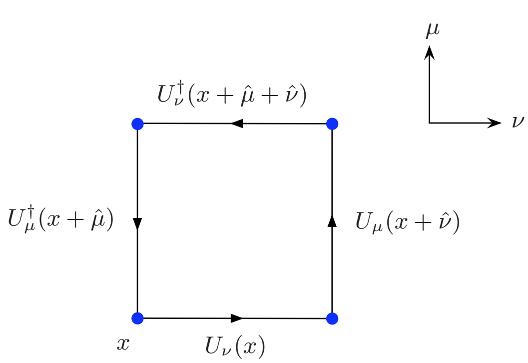

FIG. 1: A typical plaquette U_P, appearing in the gauge action (1).

Boundary conditions

(I'll clean this section up later.)

The notation in this section generally follows the review paper by Sommer [1].

Schrodinger Functional (SF): L^4 Euclidean box. Dirichlet boundaries in the time direction, periodic in spatial directions.

Non-perturbative gauge BC [2]:

$$\begin{cases} \phi_1 = -\frac{\pi}{6} & \phi_1' = -\frac{5\pi}{6} \\ \phi_2 = 0 & \phi_2' = -\frac{\pi}{3} \\ \phi_3 = \frac{\pi}{6} & \phi_3' = \frac{\pi}{2} \end{cases}$$

Running coupling gauge BC [1]:

$$\begin{cases} \phi_1 = \eta - \frac{\pi}{3} & \phi_1' = -\eta - \pi \\ \phi_2 = -\frac{1}{2}\eta & \phi_2' = \frac{1}{2}\eta + \frac{\pi}{3} \\ \phi_3 = -\frac{1}{2}\eta + \frac{\pi}{3} & \phi_3' = \frac{1}{2}\eta + \frac{2\pi}{3} \end{cases}$$

The gauge link values on the boundary are then defined by

$$C_k = \begin{pmatrix} \phi_1 & & \\ & \phi_2 & \\ & & \phi_3 \end{pmatrix},\ C_k' = \begin{pmatrix} \phi_1' & & \\ & \phi_2' & \\ & & \phi_3' \end{pmatrix}$$

and

$$\begin{cases} U_k(x)|_{t=0} & = \exp(aC_k) \\ U_k(x)|_{t=L} & = \exp(aC_k'). \end{cases}$$

Fermion spatial boundary conditions:

$$\begin{cases} \psi(x+L\hat{k}) & = e^{i\theta_k}\psi(x) \\ \overline{\psi}(x+L\hat{k}) & = \overline{\psi}(x)e^{-i\theta_k}. \end{cases}$$

Setting all $\theta_k = 0$ recovers simple periodic boundary conditions.

Various spin projections of fermions on the boundary are set equal to $\rho, \bar{\rho}, \rho', \bar{\rho}'$. These boundary values are typically taken to be zero, but fermion boundary effects still appear via functional derivatives with respect to the boundary fields,

$$\zeta(x) \equiv \frac{\delta}{\delta \bar{\rho}(x)},\ \bar{\zeta}(x) \equiv -\frac{\delta}{\delta \rho(x)}.$$

Non-perturbative tuning

We are interested in simulating at zero physical quark mass. However, with Wilson fermions the bare quark mass m_0 will be additively renormalized, i.e. it will be related to the renormalized quark mass m_R by

$$m_R = m_0 - m_0^c.$$

The physical quark mass will vanish when the bare quark mass is tuned to the critical point $m_0 = m_0^c$. In general, the value of the critical point will depend upon the characteristics of the lattice being simulated, including the coupling strength. This dependence can be strongly non-perturbative, so to determine m_0^c we will need to perform a non-perturbative tuning, which is outlined below. The improvement coefficient c_{sw} can be tuned in exactly the same way, so we can determine that along with the critical bare mass for little additional cost.

Tuning of the renormalized mass is accomplished via the partial conservation of the axial current (PCAC) relation [1]. Classically, the divergence of the axial current satisfies the operator equation

$$\partial^\mu A_\mu^a(x) = 2mP^a(x)$$

where $A_\mu^a(x)$ is the axial current $\bar{\psi}(x)\gamma_5\gamma_\mu\sigma^a\psi(x)$, while $P^a(x)$ is the pseudoscalar current, $\bar{\psi}(x)\gamma_5\sigma^a\psi(x)$. More precisely, a quantum theory satisfies the equation

$$<[\partial^\mu A_\mu^a(x) - 2mP^a(x)]\mathcal{O}_{ext}>= 0$$

where $\mathcal{O}_{ext}$ is some operator that vanishes at x. (See Sommer's review paper, arXiv:hep-lat/0611020, p.19.) Going explicitly now to the Schrodinger functional picture, we can use take this identity to define a current quark mass

$$m = \frac{<\left[\partial^\mu A_\mu^a(x)\right]\mathcal{O}^a>}{2<P^a(x)\mathcal{O}^a>}$$

where the operator $\mathcal{O}^a$ is a pseudoscalar formed from the boundary fermion fields,

$$\mathcal{O}^a = \int d^3x \int d^3y\ \bar{\zeta}(x)\gamma_5\sigma^a\zeta(y).$$

We then fine-tune the bare mass in simulation until the measured current quark mass vanishes. The clover term can also be tuned at the same time, by measuring the mass at several points in the bulk and looking for variations of order a. The exact procedure is as follows:

- Fix β and L/a, and choose initial guesses for m_0 and c_{sw}.
- Perform a simulation. Measure correlators between pseudoscalar and axial operators and the boundary fermion fields as above, to determine the current mass m.
- If $m \neq 0$, change m_0; if there is an $O(a)$ difference between values of m in the bulk, change c_{sw}.
- Repeat the two steps above until the critical point (m_0^c, c_{sw}^c) at which m vanishes to $O(a^2)$ is located.
- Repeat all of the above at a new choice of β and L/a.

The exact algorithm and procedure above is detailed in Luscher et. al., NPB 491 (1997) p.323.

Since all the tuning above is done in the bulk, it is conceivable that the L/a dependence will be weak, allowing us to perform the tuning only at small L/a but use the derived parameters for all lattice sizes. Furthermore, rather than going through the complete procedure above for each value of β that we wish to simulate at, it should be possible to perform the tuning at a range of β values and then fit the critical values to an interpolating function, obtaining $m_0^c(\beta)$ and $c_{sw}^c(\beta)$.

Perturbative improvement

The existence of Dirichlet boundaries in the SF formulation leads to the introduction of $O(a)$ lattice artifact boundary terms, in both the gauge and fermion actions. We can introduce counterterms into the action to cancel out these contributions.

Four such counterterms exist, conventionally denoted c_s, c_t, $\tilde{c}_s$, and $\tilde{c}_t$. The first two appear in the gauge action, the second two in the fermion action. The labels s and t multiply boundary terms in the action involving only spatial links on the boundary and temporal links connected to the boundary, respectively.

Neither of the purely spatial counterterms will appear in our study of the running coupling. c_s does not appear in the $O(a)$ effective action when the "running coupling" gauge boundary conditions noted above are chosen [1]. $\tilde{c}_s$ is purely a product of spatial boundary fermion fields ρ, which for our purposes are set to zero as noted above.

The coefficient c_t simply appears in front of the gauge action (1), multiplying all temporal plaquettes which intersect either of the Dirichlet boundaries. Its value is known to two-loops in lattice perturbation theory [3]:

$$c_t = 1 + (-0.08900(5) + 0.0191410(1)N_f)\, g_0^2 \\ + \left(-0.0294(3) + 0.002(1)N_f + 0.0000(1)N_f^2\right) g_0^4 + O(g_0^6).$$

The last coefficient $\tilde{c}_t$ appears with an additional term in the fermion action [4]:

$$\delta S_f = a^4 \sum_x (\tilde{c}_t - 1) \Bigg[\left(\overline{\psi}(x) \frac{1}{2}(1+\gamma^0)\nabla_0^\star \psi(x) + \overline{\psi}(x)\overleftarrow{\nabla}_0^\star \frac{1}{2}(1-\gamma^0)\psi(x) \right) \delta(x^4 - a) + \left(\overline{\psi}(x) \frac{1}{2}(1-\gamma^0)\nabla_0 \psi(x) + \overline{\psi}(x)\overleftarrow{\nabla}_0 \frac{1}{2}(1+\gamma^0)\psi(x) \right) \delta(x^4 - (L-a)) \Bigg]$$

where ∇_μ is a lattice version of a gauge-covariant derivative operator, defined in [4]. If we expand out the derivatives and set all boundary fermion fields ρ to zero, the surviving contribution of this term to the action is

$$\delta S_f|_{\rho=0} = a^4 \sum_x (\tilde{c}_t - 1) \left\{ \frac{1}{a} \overline{\psi}(x)\psi(x) \left[\delta(x^4 - a) + \delta(x^4 - (L-a)) \right] \right\}$$

which is effectively a correction to the bare quark mass term at $x^4 = a$ and $x^4 = L - a$, i.e. making the modification

$$m_0' = m_0 + (\tilde{c}_t - 1)(\delta_{t,a} + \delta_{t,L-a})$$

in the fermion action (2). The perturbative value of $\tilde{c}_t$ is known to one loop, and is independent of N_f [3]:

$$\tilde{c}_t = 1 - 0.01795(2) g_0^2 + O(g_0^4).$$

[1] R. Sommer, *Non-perturbative qcd: renormalization, o(a)-improvement and matching to heavy quark effective theory* (2006), arXiv:hep-lat/0611020.
[2] M. Luscher, S. Sint, R. Sommer, P. Weisz, and U. Wolff, Nuclear Physics B **491**, 323 (1997), hep-lat/9609035.
[3] A. Bode, P. Weisz, and U. Wolff, Nuclear Physics B **576**, 517 (2000), hep-lat/9911018.
[4] M. Luscher, S. Sint, R. Sommer, and P. Weisz, Nuclear Physics B **478**, 365 (1996), hep-lat/9605038.